Largest Hypersphere of Stability for Polynomials with Perturbed Coefficients

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Introduction

THIS study is a continuation of an earlier study[1] in which a method for finding the radius of the largest hypersphere of stability for a nominally stable system in the parameter space was introduced. Closed-form solutions to find such a radius for a second-order system were reported. In this Note we intend to apply the same method used in Ref. 1, i.e., the Lagrange multiplier approach, to Hurwitz polynomials under coefficient perturbations to obtain closed-form solutions for third- and fourth-order polynomials.

Consider the characteristic equation of a nominally stable linear system

$$f(s) = s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 \tag{1}$$

in which some or all of its coefficients b_i are subject to perturbation. Finding bounds on the perturbing coefficient of Eq. (1) is an important step for the analysis and design of many robust controllers. By a robust control system we mean that the control system remains stable for uncertain but bounded parameter variations. There are many approaches to the problem of finding perturbation bounds for Hurwitz polynomials. One approach, which is numerical in nature, requires the global minimization of a rational function.[2] Other approaches consider tests for a priori given perturbation intervals, e.g., the Kharitonov theorem.[3,4] The time-domain approach that is based on Lyapunov theory[5-7] gives conservative measures for bounds on individual parameters as well as conservative spectral norms of the system perturbation matrix.[8] In this Note we will use the Lagrange multiplier approach to obtain closed-form solutions for calculating the largest hypersphere of stability for third- and fourth-order Hurwitz polynomials. In the next section a brief description of the Lagrange multiplier approach is outlined. The main results will then follow. Illustrative examples are considered.

Preliminaries

If we start with the system description given in Eq. (1), based on Jury and Pavlidis's result,[9] the critical conditions for stability limits are given by

$$b_0 = 0 \tag{2a}$$

and

$$\Delta_{n-1} = \begin{vmatrix} b_{n-1} & b_{n-3} & b_{n-5} & b_{n-7} & \cdot & 0 \\ 1 & b_{n-2} & b_{n-4} & b_{n-6} & \cdot & \cdot \\ 0 & b_{n-1} & b_{n-3} & b_{n-5} & \cdot & \cdot \\ 0 & b_n & b_{n-2} & b_{n-4} & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdot & b_0 \\ 0 & 0 & 0 & 0 & \cdot & b_1 \end{vmatrix} = 0, \qquad n > 1 \tag{2b}$$

where $\Delta_{n-1} \in \Re^{(n-1)\times(n-1)}$. These two critical conditions for stability limits (2) can be formulated as two boundary hypersurfaces in the coefficients' space as

$$b_0 \equiv h_1 = 0 \tag{3a}$$

$$\Delta_{n-2} \equiv h_2(b_0, b_1, \cdots, b_{n-1}) = 0 \tag{3b}$$

They separate the coefficients' space into stable and unstable regions. By finding the shortest distance from the stable nominal point to the respective boundary hypersurfaces, the largest hypersphere centered at the nominal point is found in which coefficients of the characteristic equation can vary without affecting system stability. The squared distance R^2 from the nominal point to any point on a boundary hypersurface is given by

$$R^2(b_0, b_1, \ldots, b_{n-1}) = (b_0 - \alpha)^2 + (b_1 - \beta)^2 + \cdots + (b_{n-1} - \zeta)^2 \tag{4}$$

where $\alpha, \beta, \ldots, \zeta$ are the nominal values of Eq. (1). With the Lagrange multiplier approach, Eq. (4) can be minimized under constraints (3) by minimizing, separately,

$$W_1(b_0, b_1, \ldots, b_{n-1}, \lambda_1) = R^2 + \lambda_1 h_1 \tag{5a}$$

$$W_2(b_0, b_1, \ldots, b_{n-1}, \lambda_2) = R^2 + \lambda_2 h_2 \tag{5b}$$

where λ_1 and λ_2 are the Lagrange multipliers. The points on the first and second boundary surfaces that are closest to the nominal point must satisfy the following two sets of equations, respectively:

$$\begin{aligned} \partial W_1/\partial b_0 &= 0 \\ \partial W_1/\partial b_1 &= 0 \\ &\vdots \\ \partial W_1/\partial b_{n-1} &= 0 \\ \partial W_1/\partial \lambda_1 &= 0 \end{aligned} \tag{6a}$$

$$\begin{aligned} \partial W_2/\partial b_0 &= 0 \\ \partial W_2/\partial b_1 &= 0 \\ &\vdots \\ \partial W_2/\partial b_{n-1} &= 0 \\ \partial W_2/\partial \lambda_2 &= 0 \end{aligned} \tag{6b}$$

Solving for *all* roots of Eqs. (6a) and (6b), respectively, and examining these, in turn, yield the minimum squared distances $W_{1,\min}$ and $W_{2,\min}$ from the nominal point to the first and second boundary hypersurfaces, respectively. [In general, the Lagrange multiplier method applies to cases in which the coefficients of Eq. (1) can depend on each other. Here we restrict ourselves to the case of independently varying coefficients.] If b_0 is a perturbing coefficient, then Eq. (6a) is readily evaluated as $W_{1,\min} = \alpha^2$ and the squared radius of the largest hypersphere of all stable points is given by

$$R_m^2 = \min[\alpha^2, W_{2,\min}] \tag{7}$$

Otherwise, for nonperturbing b_0, we have only W_2 to consider and

$$R_m^2 = W_{2,\min} \tag{8}$$

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Consequently, the coefficients of Eq. (1) can be perturbed without affecting stability as long as

$$R_m^2 > \Delta b_0^2 + \Delta b_1^2 + \cdots + \Delta b_{n-1}^2 \tag{9}$$

which could be used to obtain bounds on $b_0, b_1, \ldots, b_{n-1}$. Obtaining bounds on the individual coefficients of Eq. (1) is referred to as highly structured perturbation. On the other hand, obtaining bounds on some norm of perturbations with no knowledge of the bounds of individual coefficients is referred to as weakly structured perturbation. Thus, R_m^2 can be used to obtain bounds for both types of perturbations.

Main Results

Even though the method is applicable to any order Hurwitz polynomials, third- and fourth-order polynomials are of particular interest because closed-form solutions can be obtained for them. Moreover, the longitudinal and lateral characteristics of aircraft can be modeled using fourth-order polynomials, e.g., see Ref. 13. For these polynomials, closed-form solutions in terms of the coefficient nominal values, i.e., α, $\beta, \ldots, \zeta$, were derived and are given in this section.

Third-Order Polynomials

Consider the third-order Hurwitz polynomial

$$f(s) = s^3 + b_2 s^2 + b_1 s + b_0 \tag{10}$$

where all coefficients vary independently. Application of the outlined method to Eq. (10) yields the largest radius of stability R_m for all possible perturbations, as given in Table 1.

Fourth-Order Polynomials

Consider the fourth-order Hurwitz polynomial

$$f(s) = s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0 \tag{11}$$

with independently varying coefficients. Application of the outlined method yields the bounds that are given in Table 2.

Examples

Example 1

Consider the strictly Hurwitz third-order polynomial used in Ref. 10

$$s^3 + 3s^2 + 3s + 3 = 0$$

When all coefficients are subject to perturbation, the last entry of Table 1 yields

$$W_i = [Z_i/(1 - Z_i^2)]\sqrt{Z_i^4 + 16Z_i^2 - 36Z_i + 19}$$

which can be simplified to

$$W_i = -[Z_i/(1 + Z_i)]\sqrt{Z_i^2 + 2Z_i + 19}$$

Z_i are the real solutions of

$$Z^5 + 3Z^4 - 2Z^3 - 15Z^2 + 19Z - 6 = 0$$

These are $\{0.5845, 1\}$. Hence,

$$R_m^2 = \min\{9, 2.7942, 5.5\} = 2.7942$$

and polynomial (11) is stable as long as $(\Delta b_0)^2 + (\Delta b_1)^2 + (\Delta b_2)^2 < 2.7942$. For the sake of comparison, the largest cube centered at $\{3,3,3\}$ that is contained in our sphere of radius $R_m = 1.67$ has a side length of 1.93, whereas the algorithm in Ref. 10 yields a value of 1.26.

Example 2

Consider the strictly Hurwitz polynomial used in Ref. 2

$$s^4 + 5s^3 + 8s^2 + 8s + 3 = 0$$

When all coefficients are subject to perturbation, the last row of Table 2 yields the two simultaneous polynomials

$$(8 - b_1)b_1 + (5 - b_3)b_3 = 0 \tag{12a}$$

$$\begin{aligned} &-(b_3^2 + 1)b_1^5 + 8(b_3 + 1)b_3 b_1^4 - 2(b_3^2 + 1)b_3^2 b_1^3 + 16 b_3^4 b_1^2 \\ &-(b_3^2 + 61)b_3^4 b_1 + 8(b_3 + 3)b_3^5 = 0 \end{aligned} \tag{12b}$$

which turns out to have three positive real solutions for b_1 and b_3. Computing the corresponding b_0 and b_2 values, we have $\{b_0, b_1, b_2, b_3\} = \{(12.1, 6.8, 12.2, 6.3), (116.9, 8.7, 210.1, 2.8), (1382.5, 8.6, 210.1, 1.3)\}$ that give rise to three W values 10.18, 118.64, and 1394.25. Consequently, the radius of the largest sphere is $R_m = 3$. [Solutions for Eqs. (12) were obtained using the software package *Mathematica*.[11] For users not having software to solve simultaneous polynomials at their disposal, the Appendix reduces the problem to that of solving a single polynomial in one variable.]

Table 1 Bound formulas of Eq. (10) ($\Delta_2^0 = \beta\gamma - \alpha$; α = nominal b_0, β = nominal b_1, and γ = nominal b_2)

Perturbation coefficient	Bound R_m^2 for $f(s) = s^3 + b_2 s^2 + b_1 s + b_0$
b_0	$R_m^2 = \min\{\alpha^2, (\Delta_2^0)^2\}$
b_1	$R_m^2 = \{(\Delta_2^0/\gamma)^2\}$
b_2	$R_m^2 = \{(\Delta_2^0/\beta)^2\}$
b_0, b_1	$R_m^2 = \min\{\alpha^2, (\Delta_2^0)^2/(1+\gamma^2)\}$
b_0, b_2	$R_m^2 = \min\{\alpha^2, (\Delta_2^0)^2/(1+\beta^2)\}$
b_1, b_2	$R_m^2 = \min\{W_i^2\}$, where $W_i = [RZ_i/(1-Z_i^2)]\sqrt{Z_i^2 - 4(\beta\gamma/R^2)Z_i + 1}$, $R^2 = \beta^2 + \gamma^2$, and Z_i are real solutions of $\alpha Z^4 - [\beta\gamma + 2\alpha]Z^2 + R^2 Z - \Delta_2^0 = 0$
b_0, b_1, b_2	$R_m^2 = \min\{\alpha^2, W_i^2\}$, where $W_i = [Z_i/(1-Z_i^2)]\sqrt{Z_i^4 + (R^2-2)Z_i^2 - 4\beta\gamma Z_i + (R^2+1)}$, $R^2 = \beta^2 + \gamma^2$, and Z_i are real solutions of $Z^5 + \alpha Z^4 - 2Z^3 - (\beta\gamma + 2\alpha)Z^2 + (R^2+1)Z - \Delta_2^0 = 0$

Table 2 Bound formulas for Eq. (11)
($\Delta_3^0 = \beta\gamma\delta - \alpha\delta^2 - \beta^2$; α = nominal b_0, β = nominal b_1, γ = nominal b_2, and δ = nominal b_3)

Perturbation coefficient	Bound R_m^2 for $f(s) = s^4 + b_3s^3 + b_2s^2 + b_1s + b_0$
b_0	$R_m^2 = \min\{\alpha^2, (\Delta_3^0/\delta)^2\}$
b_1	$R_m^2 = \min\{[\gamma\delta/2 - \beta \pm \delta\sqrt{(\gamma/2)^2 - \alpha}]^2\}$
b_2	$R_m^2 = \{(\Delta_3^0/\beta\delta)^2\}$
b_3	$R_m^2 = \min\{[(\beta\gamma/2\alpha) - \delta \pm \beta\sqrt{(\gamma/2\alpha)^2 - 1}]^2\}$
b_0, b_1	$R_m^2 = \min\{\alpha^2, W_i^2\}$, where $W_i = [Z_i/(1-2Z_i)]\sqrt{(1-2Z_i)\delta^4 + (\gamma\delta)^2 - 4\beta\gamma\delta + 4\beta^2}$ and Z_i are real solutions of $Z^2 - (1/2)[1 - (\gamma/\delta)^2 - 2\alpha/\delta^2]Z + \Delta_3^0/2\delta^4 = 0$
b_0, b_2	$R_m^2 = \min\{\alpha^2, (\Delta_3^0)^2\}$
b_0, b_3	$R_m^2 = \min\{\alpha^2, W_i^2\}$; $W_i^2 = [(b_{0i} - \alpha)^2 + (b_{3i} - \delta)^2]$ $b_{0i} = (\beta\gamma b_{3i} - \beta^2)/b_{3i}^2$, where b_{3i} are the positive real solutions of $b_3^6 - \delta b_3^5 + (\alpha\beta\gamma)b_3^3 - [\gamma^2 + 2\alpha]\beta^2 b_3^2 + 3(\beta^3\gamma)b_3 - 2\beta^4 = 0$
b_1, b_2	$R_m^2 = \min\{W_i^2\}$; $W_i^2 = [(b_{1i} - \beta)^2 + (b_{2i} - \gamma)^2]$ $b_{2i} = (\alpha\delta^2 + b_{1i}^2)/\delta b_{1i}$, where b_{1i} are the positive real solutions of $(\delta^2 + 1)b_1^4 - \delta[\beta\delta + \gamma]b_1^3 + (\alpha\gamma\delta^3)b_1 - \alpha^2\delta^4 = 0$
b_1, b_3	$R_m^2 = \min\{W_i^2\}$; $W_i^2 = [(b_{1i} - \beta)^2 + (b_{3i} - \delta)^2]$ $b_{1i} = b_{3i}[\gamma(\alpha+1)b_{3i} - (2\alpha\beta + \delta\gamma)]/\{[2(1-\alpha) + \gamma^2]b_{3i} - (2\delta + \beta\gamma)\}$, where b_{3i} are the positive real solutions of $Ab_3^2 + Bb_3 + C = 0$ with $A = -4\alpha + 8\alpha^2 - 4\alpha^3 + \gamma^2 - 6\alpha\gamma^2 + \alpha^2\gamma^2 + \gamma^4$ $B = 8\alpha\delta - 8\delta\alpha^2 + 4\alpha\beta\gamma + 4\alpha^2\beta\gamma - 2\delta\gamma^2 + 6\alpha\gamma^2\delta - \beta\gamma^3 - \alpha\beta\gamma^3 - \delta\gamma^4$ $C = -4\alpha\delta^2 - 4\alpha^2\beta^2 - 4\alpha\beta\gamma\delta + \gamma^2\delta^2 + \alpha\beta^2\gamma^2 + \beta\gamma^3\delta$
b_2, b_3	$R_m^2 = \min\{W_i^2\}$; $W_i^2 = [(b_{2i} - \gamma)^2 + b_{3i} - \delta)^2]$ $b_{2i} = (\beta b_{3i}^2 + \alpha^2)/\alpha b_{3i}$, where b_{3i} are the positive real solutions of $-(\alpha^2 + \beta^2)b_3^4 + \beta(\alpha\gamma - \beta\delta)b_3^3 - \beta^3\gamma b_3 + \beta^4 = 0$
b_0, b_1, b_2	$R_m^2 = \min\{\alpha^2, W^2\}$; $W^2 = [(b_0 - \alpha)^2 + (b_1 - \beta)^2 + (b_2 - \gamma)^2]$ $b_{0i} = b_{1i}(\delta b_{2i} - b_{1i})/\delta^2$, $b_{2i} = [b_{1i}^3 + \alpha\delta^2 b_{1i} + \gamma\delta^3]/\delta(b_{1i}^2 + \delta^2)$, where b_{1i} are the positive real solutions of $Ab_1^5 + Bb_1^4 + Cb_1^3 + Db_1^2 + Eb_1 + F = 0$ with $A = 1 + \delta^2$, $B = -(\gamma + \beta\delta)\delta$, $C = 2\delta^2(1 + \delta^2)$, $D = (\alpha\gamma - 3\gamma - 2\beta\delta)\delta^3$, $E = (2\alpha - \alpha^2 + \gamma^2 + \delta^2)\delta^4$, and $F = -(\alpha\gamma + \beta\delta)\delta^5$
b_0, b_2, b_3	$R_m^2 = \min\{\alpha^2, W_i^2\}$; $W_i^2 = [(b_{0i} - \alpha)^2 + (b_{2i} - \gamma)^2 + (b_{3i} - \delta)^2]$ $b_{0i} = (\beta b_{2i} b_{3i} - \beta^2)/b_{3i}^2)$, $b_{2i} = [\gamma b_{3i}^3 + \alpha\beta b_{3i}^2 + \beta^3]/[b_{3i}(\beta^2 + b_{3i}^3)]$, where b_{3i} are the positive real solutions of $b_3^8 - \delta b_3^7 + 2\beta^2 b_3^6 + \beta(\alpha\gamma - 2\beta\delta)b_3^5 + \beta^2(\alpha^2 + \beta^2 - \gamma^2 - 2\alpha)b_3^4$ $- \beta^3(3\gamma - \beta\delta - \alpha\gamma)b_3^3 - 2\beta^4 b_3^2 + \beta^5\gamma b_3 - \beta^6 = 0$
b_0, b_1, b_3	$R_m^2 = \min\{\alpha^2, W_i^2\}$; $W_i^2 = [(b_{0i} - \alpha)^2 + (b_{1i} - \beta)^2 + (b_{3i} - \delta)^2]$ $b_{0i} = [(\gamma b_{3i} - b_{1i})b_{1i}]/b_{3i}^2$, where (b_{3i}, b_{1i}) are the positive real solutions of $(*)\begin{cases} 2b_1^3 - 3\gamma b_3 b_1^2 + (2\alpha + \gamma^2 + b_3^2)b_3^2 b_1 - (\alpha\gamma + \beta b_3)b_3^3 = 0 \\ -2(\gamma b_3 - \beta)b_1^2 + (\gamma^2 b_3 + 2\alpha b_3 + b_3^3 + 2b_3 + \beta\gamma - 2\delta)b_3 b_1 + [\beta b_3^2 + (\alpha - 1)\gamma b_3 + \gamma\delta]b_3^2 = 0 \end{cases}$
b_1, b_2, b_3	$R_m^2 = \min\{W_i^2\}$; $W_i^2 = [(b_{1i} - \beta)^2 + (b_{2i} - \gamma)^2 + (b_{3i} - \delta)^2]$ $b_{2i} = (\alpha b_{3i}^2 - b_{1i}^2)/b_{1i} b_{3i}^2$, where (b_{3i}, b_{1i}) are the positive real solutions of $(**)\begin{cases} (\beta - b_1)b_1 + (\delta - b_3)b_3 = 0 \\ b_1^4 - \gamma b_3 b_1^3 + (\delta - b_3)b_3^3 b_1^2 + \alpha\gamma b_3^3 b_1 - \alpha^2 b_3^4 = 0 \end{cases}$
b_0, b_1, b_2, b_3	$R_m^2 = \min\{\alpha^2, W_i^2\}$; $W_i^2 = [(b_{0i} - \alpha)^2 + (b_{1i} - \beta)^2 + (b_{2i} - \gamma)^2 + (b_{3i} - \delta)^2]$ $b_{0i} = (b_{2i} b_{3i} - b_{1i})b_{1i}/b_{3i}^2, b_{2i} = [b_{1i}^3 + \alpha b_{1i} b_{3i}^2 + \gamma b_{3i}^3]/[b_{3i}(1 + b_{3i}^2)]$, where (b_{3i}, b_{1i}) are the positive real solutions of $(***)\begin{cases} (\beta - b_1)b_1 + (\delta - b_3) = 0 \\ Ab_1^5 + Bb_1^4 + Cb_1^3 + Db_1^2 + Eb_1 + F = 0 \end{cases}$ $A = -(b_3^2 + 1)$, $B = (\gamma + \beta b_3)b_3$, $C = -2(1 + b_3^2)b_3^2$, $D = (2\beta b_3 + 3\gamma - \alpha\gamma)b_3^3$, $E = (\alpha^2 - \gamma^2 - 2\alpha - b_3^2)b_3^4$, and $F = (\alpha\gamma + \beta b_3)b_3^5$

Alternatively, systems (*), (**), and (***) can be solved as single polynomials given in the Appendix.

Conclusion

Closed-form solutions for finding the largest possible hypersphere of stability for third- and fourth-order Hurwitz polynomials are reported. These formulas are very convenient to use and reduce computational effort to a minimum. They can be employed to find bounds under weakly structured perturbations as well as highly structured perturbations.

Appendix

The three sets of simultaneous polynomials in Table 2 are reduced to single polynomials that are shown here. Since a special form of the Euclidean algorithm[12] is utilized, the resultant polynomials have more roots than the original two polynomials. These extra roots can be easily identified and eliminated by back substitution in the original polynomials. These equations were derived with the help of *Mathematica*[11] and verified numerically against their corresponding equations in Table 2.

Equation (*) can be solved by solving:

$$\begin{aligned}&(\gamma \boldsymbol{b}_3 - 2\beta)(\gamma \boldsymbol{b}_3 - \beta)[\boldsymbol{b}_3^3 + 2(\alpha - \gamma^2 - 1)\boldsymbol{b}_3 + 2(\delta + \beta\gamma)][\boldsymbol{b}_3^7 - \delta \boldsymbol{b}_3^6 + 2(2\alpha + \gamma^2 - 2)\boldsymbol{b}_3^5 + (8\delta - 4\alpha\delta - 9\beta\gamma + \alpha\beta\gamma - 2\delta\gamma^2)\boldsymbol{b}_3^4 + (4 - 8\alpha\\
&\quad + 4\alpha^2 + 8\beta^2 - 2\alpha\beta^2 - 4\delta^2 + 9\beta\gamma\delta + 5\gamma^2 - 2\alpha\gamma^2 - \beta^2\gamma^2 + \alpha^2\gamma^2 + \gamma^4)\boldsymbol{b}_3^3 + (16\alpha\delta - 12\delta - 4\alpha^2\delta - 8\beta^2\delta - 6\beta\gamma - 2\alpha^2\beta\gamma - 3\beta^3\gamma\\
&\quad - 10\delta\gamma^2 + 2\alpha\delta\gamma^2 - 3\beta\gamma^3 - \alpha\beta\gamma^3 - \delta\gamma^4)\boldsymbol{b}_3^2 + (4\alpha\beta^2 - 2\beta^4 + 12\delta^2 + 12\delta^2 - 8\alpha\delta^2 + 2\beta^2\gamma^2 + 3\alpha\beta^2\gamma^2 + 5\delta^2\gamma^2 - 3\beta\delta\gamma^3 + 12\beta\gamma\delta)\boldsymbol{b}_3\\
&\quad - (2\alpha\beta^3\gamma + 4\alpha\beta^2\delta + 2\beta^2\delta\gamma^2 + 6\beta\gamma\delta^2 + 4\delta^3)] = 0\end{aligned}$$

Equation (**) can be solved by solving:

$$\begin{aligned}&\boldsymbol{b}_3^8 - 4\delta \boldsymbol{b}_3^7 + 2(3\delta^2 - \alpha^2 + 1)\boldsymbol{b}_3^6 + (\beta\gamma - \alpha\beta\gamma - 8\delta - 4\delta^3 + 4\alpha^2\delta)\boldsymbol{b}_3^5 + (1 - 2\alpha^2 + \alpha^4 - \beta^2 + \alpha^2\beta^2 + \gamma^2 + 2\alpha\gamma^2 + \alpha^2\gamma^2 - 3\beta\gamma\delta\\
&\quad + 2\alpha\beta\gamma\delta + 12\delta^2 - 2\alpha^2\delta^2 + \delta^4)\boldsymbol{b}_3^4 + (-4\delta - \beta\gamma - 3\alpha\beta\gamma - 3\alpha^2\beta\gamma - \alpha^3\beta\gamma + 4\alpha^2\delta + 3\beta^2\delta - \alpha^2\beta^2\delta - 3\gamma^2\delta - 4\alpha\gamma^2\delta - \alpha^2\gamma^2\delta + 3\beta\gamma\delta^2\\
&\quad - \alpha\beta\gamma\delta^2 - 8\delta^3)\boldsymbol{b}_3^3 + (4\alpha^2\beta^2 - \alpha\beta^2\gamma^2 + 3\beta\gamma\delta + 6\alpha\beta\gamma\delta + 3\alpha^2\beta\gamma\delta + 6\delta^2 - 2\alpha^2\delta^2 - 3\beta^2\delta^2 + 3\gamma^2\delta^2 + 2\alpha\gamma^2\delta^2 - \beta\gamma\delta^3 + 2\delta^4)\boldsymbol{b}_3^2 + (\alpha\beta^3\gamma\\
&\quad + \alpha^2\beta^3\gamma - 4\alpha^2\beta^2\delta + \alpha\beta^2\gamma^2\delta - 3\beta\gamma\delta^2 - 3\alpha\beta\gamma\delta^2 - 4\delta^3 + \beta^2\delta^3 - \gamma^2\delta^3)\boldsymbol{b}_3 + (\beta\gamma\delta^3 + \delta^4 - \alpha\beta^3\gamma\delta - \alpha^2\beta^4) = 0\end{aligned}$$

Equation (***) can be solved by solving:

$$\begin{aligned}&[\beta \boldsymbol{b}_3^4 + (2\gamma - \alpha\gamma - \beta\delta)\boldsymbol{b}_3^3 + \gamma\delta \boldsymbol{b}_3^2 + \beta(\beta\gamma - 2\gamma)\boldsymbol{b}_3 - \beta^3][A\boldsymbol{b}_3^5 - \delta A\boldsymbol{b}_3^4 + (2\beta^2 - 4\alpha\beta^2 + 2\alpha^2\beta^2 + 2\beta^4 + \beta^2\gamma^2 - 2\alpha\beta^2\gamma^2 - \alpha^2\beta^2\gamma^2\\
&\quad - 5\beta\gamma\delta + 3\alpha\beta\gamma\delta + \alpha^2\beta\gamma\delta + \alpha^3\beta\gamma\delta + \beta^3\gamma\delta - \alpha\beta^3\gamma\delta - 3\beta\gamma^3\delta - \alpha\beta\gamma^3\delta - 2\delta^2 + 4\alpha\delta^2 - 2\alpha^2\delta^2 + 5\beta^2\delta^2 - 2\alpha\beta^2\delta^2 + \alpha^2\beta^2\delta^2 - \gamma^2\delta^2\\
&\quad + 2\alpha\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 - \beta^2\gamma^2\delta^2 - \beta\gamma\delta^3 + \alpha\beta\gamma\delta^3 + 2\delta^4)\boldsymbol{b}_3^3 + (-3\beta^3\gamma + 2\alpha\beta^3\gamma + \alpha^2\beta^3\gamma + \beta^5\gamma - \beta^3\gamma^3 - 6\beta^2\delta + 12\alpha\beta^2\delta - 6\alpha^2\beta^2\delta\\
&\quad - 2\beta^4\delta + 6\alpha\beta^2\gamma^2\delta + 9\beta\gamma\delta^2 - 6\alpha\beta\gamma\delta^2 - 3\alpha^2\beta\gamma\delta^2 + 2\beta^3\gamma\delta^2 + 3\beta\gamma^3\delta^2 + 2\delta^3 - 4\alpha\delta^3 + 2\alpha^2\gamma^3 - 4\beta^2\delta^3 - 2\alpha\gamma^2\delta^3 + \beta\gamma\delta^4 - 2\delta^5)\boldsymbol{b}_3^2\\
&\quad + (2\alpha\beta^4 - \alpha^2\beta^4 - \beta^6 + \beta^4\delta^2 + \alpha\beta^4\delta^2 + 6\beta^3\gamma\delta - 5\alpha\beta^3\gamma\delta + \alpha^2\beta^3\gamma\delta + \beta^3\gamma^3\delta + 6\beta^2\delta^2 - 8\alpha\beta^2\delta^2 + 4\alpha^2\beta^2\delta^2 - 2\beta^4\delta^2 - \beta^2\gamma^2\delta^2 - \alpha\beta^2\gamma^2\delta^2\\
&\quad - 3\beta\gamma\delta^3 + 3\alpha\beta\gamma\delta^3 + \delta^4 - \beta^2\delta^4 + \gamma^2\delta^4)\boldsymbol{b}_3 + (-\alpha\beta^5\gamma - 2\alpha\beta^4\delta + \alpha^2\beta^4\delta - \beta^4\gamma^2\delta - 3\beta^3\gamma\delta^2 + \alpha\beta^3\gamma\delta^2 - 2\beta^2\delta^3 - \beta\gamma\delta^4 - \delta^5)] = 0;\ A\\
&\quad = (1 - 4\alpha + 6\alpha^2 - 4\alpha^3 + \alpha^4 + 2\beta^2 - 4\alpha\beta^2 + 2\alpha^2\beta^2 + \beta^4 + 2\gamma^2 - 4\alpha\gamma^2 + 2\alpha^2\gamma^2 - 2\beta^2\gamma^2 + \gamma^4 - 8\beta\gamma\delta + 8\alpha\beta\gamma\delta - 2\delta^2 + 4\alpha\delta^2\\
&\quad - 2\alpha^2\delta^2 + 2\beta^2\delta^2 + 2\gamma^2\delta^2 + \delta^4)\end{aligned}$$

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New Analytical Solutions for Proportional Navigation

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I. Introduction

PROPORTIONAL navigation (PN) is accepted as a celebrated guidance law for many guided missile applications like the surface-to-air, air-to-air, and air-to-surface missile encounters, standoff weapon delivery, and space rendezvous. In earlier studies, the guided point (pursuer) is assumed to move toward a target point in a plane containing the velocity vectors of the two points.[1] The PN strategy is defined such that the velocity vector (heading) of the pursuer is turned at a rate proportional to the rotation rate of the the line joining the pursuer and target which is the line of sight (LOS). The PN principle helps to estimate the magnitude of the lateral acceleration as a function of LOS turn rate. Three different types of PN have been defined: pure proportional navigation (PPN), true proportional navigation (TPN), and generalized true proportional navigation (GTPN) (see Fig. 1).[7]

All cases of PN are described by highly nonlinear equations of motion. Currently, there is a great deal of interest in obtain-

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